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Plane Sections of a Weddle Surface.

By F. Morley and J. R. Conner.

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§1. *An Equation of the Surface.*

The Jacobian of all quadrics on six given points of space is a Weddle quartic surface W; that is to say, W is the locus of points x, y apolar to all quadrics on the 6 points.

We use the symbol :: for "is or are apolar to." The 6 points determine a norm-curve N, which may be given by

$$x_0 = 1,\ x_1 = -t,\ x_2 = t^2,\ x_3 = -t^3. \tag{1}$$

The points are then defined by a sextic equation

$$\alpha \equiv (\alpha t)^6 = 0.$$

The quadrics on the 6 points are a 3-fold system, built on three containing the points of N, namely,

$$x_0x_2 = x_1^2,\ x_0x_3 = x_1x_2,\ x_1x_3 = x_2^2,$$

and one other not containing the points of N. That other is chosen which :: all quadrics on the planes of N. We speak of this as a quadric (of points) *orthic* to N. It is uniquely defined when the sextic α is given and may be written $(\alpha x)^2$.

The quadrics which :: all quadrics on the 6 points are then quadrics (of planes) which are orthic to N and also :: $(\alpha x)^2$.

Now if we have a quadric of points $(\alpha x)^2$ and a quadric of planes $(a\xi)^2$, both orthic to N, then it is to be shown that the binary sextic defining the points common to α and N :: the sextic defining the planes common to a and N. This we shall prove analytically. If $(a\xi)^2$ be orthic, then

$$a_{02} = a_{11},\ a_{03} = a_{12},\ a_{13} = a_{22}. \tag{2}$$

The binary sextics are

$$(a_0t^3 + 3a_1t^2 + 3a_2t + a_3)^2,\ (\alpha_0 - \alpha_1t + \alpha_2t^2 - \alpha_3t^3)^2.$$

Bearing (2) in mind, the apolarity-condition of these sextics is

$$a_{00}\alpha_{00} + 2a_{01}\alpha_{01} + a_{11}(\alpha_1^2 + 2\alpha_{02}) + 2a_{12}(\alpha_{12} + \alpha_{03}) + a_{22}(\alpha_2^2 + 2\alpha_{13}) + 2a_{23}\alpha_{23} + a_{33}\alpha_{33} = 0,$$

or

$$(a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)^2 = 0.$$

That is, the sextics are apolar if the quadrics are, and conversely.*

We have now a convenient definition of a Weddle surface, when the sextic is given on N. It is the locus of points x, y representing† binary cubics $(xt)^3$, $(yt)^3$, subject to the conditions, first, that they are an orthic quadric of N, *i. e.*, that

$$\left.\begin{aligned} x_0y_2 + x_2y_0 &= 2x_1y_1, \\ x_0y_3 + x_3y_0 &= x_1y_2 + x_2y_1, \\ x_1y_3 + x_3y_1 &= 2x_2y_2, \end{aligned}\right\} \tag{3}$$

each cubic being now the Jacobian of the other; second, that $(xt)^3(yt)^3 :: (\alpha t)^6$, *i. e.*, that

$$|\alpha x|^3 |\alpha y|^3 = 0. \tag{4}$$

This equation (4) we regard as the equation of W. If from (3) and (4) we eliminate y, we have the usual form for W (Baker: Multiply Periodic Functions, p. 65).

§ 2. *Configurations on the Section.*

Written out, (4) is

$$\alpha_6x_0y_0 - 3\alpha_5(x_0y_1 + x_1y_0) + 15\alpha_4x_1y_1 - 10\alpha_3(x_0y_3 + x_3y_0) + 15\alpha_2x_2y_2 - 3\alpha_1(x_2y_3 + x_3y_2) + \alpha_0x_3y_3 = 0.$$

Now if in particular the 6 points coincide by twos, $(\alpha t)^6$ may be taken as

$$\lambda t^6 + 2\lambda t^3 + \lambda = 0,$$

and (4) becomes

$$\lambda(x_0y_0 - x_0y_3 - x_3y_0 + x_3y_3) = 0,$$

and this special Weddle surface breaks up into the plane $x_0 = x_3$ which joins the 3 points, and the cubic surface into which this plane is transformed by (3).‡

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* It is only required that *one* of the quadrics be orthic. The theorem seems to be a case of Study's general theory, *Math. Annalen*, Vol. XL.

† We write here the coördinates of a point x as x_{000}, x_{001}, x_{011}, x_{111}.

‡ The cubic will be briefly characterized later (§ 6).

Since (4) is linear in the coefficients α, a pencil of sextics gives a pencil of Weddle surfaces, and thus in particular the pencil of sextics

$$(\alpha t)^6 + \lambda(t^3 + 1)^2 = 0$$

gives *a pencil of surfaces* W_λ *having a common plane section* Q.

The 6 original points, 1, ..., 6, are nodes of W_0; their joins, 12, ..., are lines of W_0; and the 10 lines $123|456$ are also lines of W_0.

These lines meet the section Q in 15 points 12, 13, ..., lying by threes on 20 lines 123, ..., and in 10 points $123|456$, the whole forming a configuration, let us say Γ.*

The curve Q contains then a configuration Γ, but being common to all the surfaces W_λ it contains infinitely many such configurations. This is the *quod erat demonstrandum;* namely, that *the plane section of a Weddle surface contains infinitely many configurations* Γ. We speak of these as a pencil of configurations.

§ 3. *The Invariant Condition.*

The existence of a configuration on a curve is apt to imply that an invariant of the curve vanishes. In the present case, the equation of Q, referred to the points e_ι where it meets N as a reference triangle, can be written at once. The curve is on the reference points; it cuts each reference line again in points apolar with the other reference lines, because the Weddle surface is invariant under the transformation (3); and also the tangents at the reference points are on a point, from the known fact that N is an asymptotic curve on W. Thus, using Salmon's notation for quartic curves and their invariants, Q is

$$Q \equiv a_2x^3y + a_3x^3z + b_3y^3z + b_1y^3x + c_1z^3x + c_2z^3y + 3xyz(lx + my + nz) = 0,$$

where

$$a_2b_3c_1 + a_3b_1c_2 = 0.$$

Hence the invariants A and B are

$$A = -12lmn + 12(lb_1c_1 + mc_2a_2 + na_3b_3),$$

$$B = -\begin{vmatrix} l & a_3 & a_2 \\ b_3 & m & b_1 \\ c_2 & c_1 & n \end{vmatrix}^2;$$

and therefore the invariant which vanishes is

$$A^2 + 144B.$$

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*The presumption is that a quartic curve on the 15 point of Γ is unique and must therefore contain also the 10 adjoined points. This, however, has not been proved.

The vanishing of this invariant is characteristic for the plane section of a Weddle surface.

§ 4. *The Locus of Lines of the Configurations.*

We shall give a second proof of the existence of configurations on the curve, and thereby establish the locus of the lines of a pencil of configurations.

Quadrics on the 6 points meet any plane π in conics apolar to a range, Φ. Quadrics on 5 of the points, say 1, 2, 3, 4, 5, meet π in conics apolar to a definite conic C_6 — the conic defining the polarity of the Desargues configuration determined on π by 1, 2, 3, 4, 5. The six conics C_i are in the range Φ.* N carries a double infinity of quadrics on the 6 points, and hence the three points p, q, r, where N meets π, carry a double infinity of conics apolar to Φ; pqr is then the diagonal 3-point of Φ.

A quadric of the system on the 6 points may be made to touch π at any given point. The two lines, ξ, η, cut out of this quadric by $\pi :: \Phi$. In particular the two generators cut by π out of a quadric cone with vertex on $\pi :: \Phi$. There is one further condition on this pair of lines: it is that *they touch a conic of the range, say* Φ', *on any four lines of* Γ *like* 123, 124, 134, 234. For the cone determines a tetrahedral complex with 1234 as fundamental tetrahedron, and the six lines touch the complex conic on π.

We state this as a

Lemma: If a tedrahedron is inscribed in a quadric cone, any plane on the vertex meets the cone and the planes of the tetrahedron in 6 lines on a conic.

The ranges Φ and Φ' are not independent; a conic of Φ' touches the diagonal lines qr, rp, pq of Φ. For N is projected from p by a quadric cone on 1234 with vertex p, and by the lemma pq and pr touch a conic of Φ'. Similarly, rp and rq touch a conic of Φ', necessarily the same conic.

Conversely, two ranges, Φ and Φ', connected in this way, determine uniquely a configuration Γ. For choose any tetrahedron with faces on the base lines of Φ'. It easily follows from the lemma that the four vertices of this tetrahedron are on a norm-curve N with p, q, r. The point 56, into which the conic of Φ' touching qr, rp, pq is transformed by the line-transformation $\xi\eta :: \Phi$, carries a unique line bisecant to N, thus determining the points 5 and 6 on N, and thereby a configuration Γ with the associated range Φ.

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* Carver, *Trans. Am. Math. Soc.*, Vol. VI, No. 4, p. 543.

We now find the locus of pairs of lines touching conics of Φ' and apolar to Φ. Choose, as the range Φ,

$$m_0x_0^2 + m_1x_1^2 + m_2x_2^2 = 0,$$
$$m_0 + m_1 + m_2 = 0;$$

pqr is then the reference triangle, and lines ξ, η apolar to Φ are connected by the relations

$$\xi_i = 1/\eta_i \qquad (i = 0, 1, 2).$$

If Φ' is $(\alpha\xi)^2 + \lambda(\beta\xi)^2 = 0$, the equation of our locus is at once

$$\Sigma \equiv \begin{vmatrix} (\alpha\xi)^2 & (\beta\xi)^2 \\ (\alpha/\xi)^2 & (\beta/\xi)^2 \end{vmatrix} = 0,$$

where $(\alpha/\xi) \equiv \alpha_0/\xi_0 + \alpha_1/\xi_1 + \alpha_2/\xi_2$. Using coördinates ξ, η, ζ for convenience of writing, and supposing Φ' to be

$$\lambda(a\eta\zeta + b\zeta\xi + c\xi\eta) + l\xi^2 + m\eta^2 + n\zeta^2 + f\eta\zeta + g\zeta\xi + h\xi\eta = 0, \tag{5}$$

we have

$$\begin{aligned}\Sigma \equiv\ & al\eta\zeta(\eta^2\zeta^2 - \xi^4) + bm\zeta\xi(\zeta^2\xi^2 - \eta^4) + cn\xi\eta(\xi^2\eta^2 - \zeta^4) \\ &+ [a(m - n) + bh - cg]\xi^2\eta\zeta(\zeta^2 - \eta^2) \\ &+ [b(n - l) + cf - ah]\eta^2\zeta\xi(\xi^2 - \zeta^2) \\ &+ [c(l - m) + ag - bf]\zeta^2\xi\eta(\eta^2 - \xi^2) = 0.\end{aligned} \tag{6}$$

We shall show that if Σ is given in the form

$$\begin{aligned}\Sigma \equiv\ & A\eta\zeta(\eta^2\zeta^2 - \xi^4) + B\zeta\xi(\zeta^2\xi^2 - \eta^4) + C\xi\eta(\xi^2\eta^2 - \zeta^4) \\ &+ F\xi^2\eta\zeta(\zeta^2 - \xi^2) + G\eta^2\zeta\xi(\xi^2 - \zeta^2) + H\zeta^2\xi\eta(\eta^2 - \xi^2) = 0,\end{aligned} \tag{7}$$

the pencil Φ' is not unique, but may be chosen in ∞^1 ways. We first find the equation of the section of W which is the locus of joins of corresponding lines of Σ. Let (α, β, γ) be a line of Σ. The join of this line and its correspondent is

$$x = \alpha(\gamma^2 - \beta^2), \quad y = \beta(\alpha^2 - \gamma^2), \quad z = \gamma(\beta^2 - \alpha^2).$$

Substituting α, β, γ in (7) and dividing by $\alpha\beta\gamma$, we have

$$\begin{aligned}&\frac{A(\beta^2\gamma^2 - \alpha^4)}{\alpha} + \frac{B(\gamma^2\alpha^2 - \beta^4)}{\beta} + \frac{C(\alpha^2\beta^2 - \gamma^4)}{\gamma} \\ &\qquad + F\alpha(\gamma^2 - \beta^2) + G\beta(\alpha^2 - \gamma^2) + H\gamma(\beta^2 - \alpha^2) = 0.\end{aligned} \tag{8}$$

Now

$$\frac{\beta^2\gamma^2 - \alpha^4}{\alpha} = \frac{y^2 - z^2}{x}, \quad \frac{\gamma^2\alpha^2 - \beta^4}{\beta} = \frac{z^2 - x^2}{y}, \quad \frac{\alpha^2\beta^2 - \gamma^4}{\gamma} = \frac{x^2 - y^2}{z}.$$

Hence (8) becomes

$$Q \equiv Ayz(y^2 - z^2) + Bzx(z^2 - x^2) + Cxy(x^2 - y^2) + xyz(Fx + Gy + Hz) = 0. \tag{9}$$

Given an equation of the form (7), we have six equations to determine $a, b, c, l, m, n, f, g, h$:

$$al = \rho A, \quad bm = \rho B, \quad cn = \rho C, \tag{10a}$$

$$\left.\begin{aligned} a(m-n) + bh - cg &= \rho F, \\ b(n-l) + cf - ah &= \rho G, \\ c(l-m) + ag - bf &= \rho H. \end{aligned}\right\} \tag{10b}$$

Multiply the first of equations (10b) by a, the second by b, the third by c, and add; then substitute the values of l, m, n from (10a). We have

$$Abc(b^2-c^2) + Bca(c^2-a^2) + Cab(a^2-b^2) + abc(Fa + Gb + Hc) = 0.$$

This shows that the conic $a\eta\zeta + b\zeta\xi + c\xi\eta = 0$ must be the transform by Φ of a point of Q. If a, b, c are chosen to satisfy (9), we may choose arbitrarily any one of the quantities f, g, h, and the other two are determined in such a manner as to satisfy all the equations (10). The arbitrariness in the choice of f, g, h may be accounted for by the fact that (6) is unaltered if we put

$$f = f' + \lambda a, \quad g = g' + \lambda g, \quad h = h' + \lambda c,$$

which means merely that (5) is any conic of Φ'.

Σ may then be generated in ∞^1 ways; a point (a, b, c) of Q determines uniquely a range Φ', and thereby a configuration Γ circumscribed to Σ and inscribed to Q.

The form of equation (7) shows that not only are the reference lines double lines of Σ, but the reference points are double points, the tangents at the double point being the two reference lines.

It is worthy of note that the range Φ is the same for all configurations of the pencil.

From the form of equation (9) we infer that any triangle inscribed in N meets W in the six vertices of a 4-line. In other words, *the join of points of W on two sides of the triangle meets W on the third side.* Given W, a configuration inscribed in Q may be determined as follows:

Choose any point 1 of N. From 1 N is projected by a quadric cone which meets Q in p, q, r, and in five other points, determining on N five points 2, 3, 4, 5, 6, and on Q five points 12, 13, 14, 15, 16. It follows from the above remark that the configuration determined by the points 1, 2, 3, 4, 5, 6 is inscribed in Q.

To construct W, when a configuration Γ is given, we may take any line on 12, and on it any two points, as 1 and 2. The points 3, 4, 5, 6 are then determined, and thereby W. For a given Γ there are then ∞^4 W's; and therefore for a given Q there are, as there should be, ∞^5 W's.

§ 5. *The Number of Pencils of Configurations.*

There will be, on the quartic curve Q, not one but several pencils of configurations. The number of pencils has not been found, nor has the preliminary question, in how many ways a quartic can be written in the form (9), that is, without even powers, been answered. But it is worth while to establish a special case, that for the Clebsch quartic, where $B = 0$, the latter number is 8. For when the equation of a quartic is thrown into the form (9), B becomes a perfect square, say $-\Delta^2$. The bordered catalecticant is

$$\left|\begin{array}{ccccccc} 0 & 0 & 0 & & & & \xi^2 \\ 0 & 0 & 0 & & \Delta & & \eta^2 \\ 0 & 0 & 0 & & & & \zeta^2 \\ & & & 0 & n & m & \eta\zeta \\ & \Delta & & n & 0 & l & \zeta\xi \\ & & & m & l & 0 & \xi\eta \\ \xi^2 & \eta^2 & \zeta^2 & \eta\zeta & \zeta\xi & \xi\eta & 0 \end{array}\right| ,$$

and when developed, every term except the even ones has Δ as a factor. Hence, when $\Delta = 0$, the $C_{5,4}$ consists of even terms. But the $C_{5,4}$ is then the square of the apolar conic.

Hence, for a Clebsch quartic each such triangle is apolar to the conic, and each such reference line is on the reciprocal of the quartic as to the conic. But the lines already touch a $C_{3,0,6}$, the locus of lines cut harmonically. There are then 24 such lines or 8 such triangles. Any two give 6 points on a conic; we have then 28 conics which should be connected with the double lines of the curve.

If our curve Q have $B = 0$, that is, if both A and B are 0, it follows that, when it is referred to any one of the 8 triangles,

$$a_2b_3c_1 + a_3b_1c_2 = 0.$$

The quartic can then be reduced in 8 ways to the form (9), and for each such form there is a definite curve Σ and a pencil of configurations (§ 4). Thus, for a curve Q with $B = 0$ there are 8 pencils of configurations Γ. This is probably true when $B \neq 0$.

§ 6. *The Weddle Cubic.*

The degenerate case when W is a plane and a cubic is of interest. The cubic is the transform of a plane by the transformation (3).

Now this transformation sends a line into a norm-curve N'. But if the line is on a point p of N, the line p of N is part of the transform, and if the line is also on the plane p of N, the line p counts twice,* and the rest of the transform is a line. Since (3) is involutory, this line is also on a point and corresponding plane of N. Let us call such a line, which transforms into a line without being a fixed line, a *permanent* line.

A plane π passes into a cubic surface Ω with nodes at the points p_ι on π and N.

In the plane π is a curve R^4 of class 3, order 4, the section of the developable of N. Curves of order 3 on π, on the cusps of R and also on the cusp-tangents, touch the planes p_ι of N at the 3 cusps, and therefore their transforms are also plane cubics, sections of Ω. In this case, then, the usual mapping of a plane on a cubic surface by a 3-fold system of cubics is merely the transformation (3).† The cusp-lines, being permanent lines, in the plane π and therefore on its null-point p, are transformed into permanent lines also on a plane π' and its null-point p', p' being the transform of p. These are lines of Ω.

Thus Ω is a cubic surface with 3 nodes, and 3 lines *in a pencil.*

Any cubic surface with a norm-curve as asymptotic curve is of this kind. For the developable occurs twice in its transform, leaving a plane.

We may notice that the transforms of sections of Ω by tangent planes at points of N are *osculants* of the R^4, for, by a theorem of Mr. Thomsen, these osculants have their nodes on the R^4.

The condition that a cubic surface have 3 lines in a pencil is the vanishing of Salmon's invariant I_{100}. For if 3 lines are in a pencil, the polar quadric of the vertex p is two planes, so that p is a double point of the Hessian. But if the point $x_3 = x_4 = x_5$ is on the surface $(\chi x^3) = 0$, then $\chi_1 = \chi_2$; that is, $I_{100} = 0$.

BALTIMORE, *January*, 1909.

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* Schoute, *Niew Archief*, 1899, p. 97.

† The general mapping by plane cubics on 6 points is effected by the transformation xy :: net of quadrics. Conner, *Johns Hopkins Circulars*, July, 1908, p. 95.